ON AN INTEGRAL FUNCTIONAL INEQUALITY

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ABSTRACT. In this paper, we establish upper bounds for the solutions of some functional integral inequalities.

1. INTRODUCTION

In a recent paper Pachpate [1] has obtained upper bounds for the solutions of the following integral inequalities

$$\text{(A)} \qquad x^2(t) \le c^2 + 2\int_0^t [f(s)x(\sigma(s))W(x(\sigma(s))) + h(s)x(\sigma(s))]\,ds,$$

$$\text{(B)} \quad x^2(t) \le c^2 + 2\int_0^t \left[f(s)x(\sigma(s))\left(\int_0^s g(\tau)W(x(\sigma(\tau)))d\tau\right) + h(s)x(\sigma(s))\right] ds,$$

(L)

$$x^2(t) \le c^2 + 2\int_0^t \left[f(s)x(\sigma(s))\left(\int_0^s g(\tau)W(logx(\sigma(\tau)))d\tau\right) + h(s)x(\sigma(s))\right] ds,$$

for $t \in [0,\infty)$, with the conditions

$$x(t) = \psi(t) \le c, t \in [\min_{t\in\mathbb{R}+} \sigma(t), 0],$$

and

$$\sigma \in C([0,\infty),\mathbb{R}), \quad \text{with} \quad \sigma(t) \le t, t \in [0,\infty).$$

The purpose of this note is to obtain upper bounds for the solutions of more general integral inequalities of the following form

$$(A_1) \qquad x^n(t) \le c^n + n\int_0^t [f(s)x^m(\sigma_1(s))W(x^r(\sigma_2(s))) + h(s)x^l(\sigma_3(s))]\,ds,$$

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$$n > r > 0, \quad 0 < m, l \leq n - r,$$

(B_1)

$$x^n(t) \leq c^n + n \int_0^t \left[f(s) x^m(\sigma_1(s)) \left(\int_0^s g(\tau) W(x^r(\sigma_2(\tau))) d\tau \right) + h(s) x^l(\sigma_3(s)) \right] ds,$$

$$n > r > 0, \quad 0 < m, l \leq n - r,$$

(L_1)

$$x^n(t) \leq c^n + n \int_0^t \left[f(s) x^m(\sigma_1(s)) \left(\int_0^s g(\tau) W(log x(\sigma_2(\tau))) d\tau \right) + h(s) x^l(\sigma_3(s)) \right] ds,$$

$$n > 0, \quad 0 < m, l \leq n,$$

where $t \in [0, \infty)$.
Also, in all above cases we suppose that

(C) $x(t) = \psi(t) \leq c, t \in [a, 0]$, where ψ is a given real function defined on $[a, 0]$ and $a = \min_{i=1,2,3} \{\min \sigma_i(t) : t \in [0, \infty)\}$.

2. Main Result

For our convenience we list bellow the assumptions we will use in the next theorem.

(H_1) $\sigma_i \in C([0, \infty), R)$, with $\sigma_i(t) \leq t, \quad t \in [0, \infty), i = 1, 2, 3$.
(H_2) $f \in C([0, \infty), [0, \infty))$.
(H_3) $h \in C([0, \infty), [0, \infty))$.
(H_4) $g \in C([0, \infty), [0, \infty))$.
(H_5) $x \in C([a, \infty), [x_0, \infty)), x_0 \geq 0, c \geq 1$.
(H_6) $x \in C([a, \infty), [x_0, \infty)), x_0 \geq 1, c \geq 1$.
(H_7) $W \in C([0, \infty), [0, \infty))$ is nondecreasing, $W(x) \geq 0$, $x > x_0$ and $W(x_0) = 0$.

Theorem. *(i) Inequality (A_1) with (C) and assumptions $(H_i), i = 1, 2, 3, 5, 7$ imply*

(1) $$x(t) \leq \left[G^{-1} \left[G \left(c^r + r \int_0^t h(s) ds \right) + r \int_0^t f(s) ds \right] \right]^{\frac{1}{r}}, 0 \leq t \leq \beta_1.$$

(ii) Inequality (B_1) with (C) and assumptions $(H_i), i = 1, 2, 3, 4, 5, 7$ imply

(2)

$$x(t) \leq \left[G^{-1} \left[G \left(c^r + r \int_0^t h(s) ds \right) + r \int_0^t f(s) \left(\int_0^s g(\tau) d\tau \right) ds \right] \right]^{\frac{1}{r}}, 0 \leq t \leq \beta_2.$$

(iii) Inequality (L_1) with (C) and assumptions $(H_i), i = 1, 2, 3, 4, 6, 7$ imply

(3)

$$x(t) \leq exp\left[G^{-1}\left[G\left(logc+\int_0^t h(s)ds\right)+\int_0^t f(s)\left(\int_0^s g(\tau)d\tau\right)ds\right]\right], 0 \leq t \leq \beta_3,$$

where

$$G(u)=\int_0^u \frac{ds}{W(s)}, u \geq u_0 > x_0,$$

G^{-1} is the inverse of G and the numbers $\beta_i, i=1,2,3$ are choosen so that the quantities in the square bracets in (1), (2), (3) are in the range of G.

Proof. In the following we give in details the proof of (ii) and (iii). The proof of the assertion (i) can be done easily following the proof of (ii).

(ii) We define a function $u\colon [0,\infty) \to [0,\infty)$, by

$$u^n(t)=(c+\varepsilon)^n+n\int_0^t \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau\right)+h(s)x^l(\sigma_3(s))\right]ds,$$

where $\varepsilon > 0$, is an arbitrary constant. Then we have

$$u(0)=c+\varepsilon$$

and

(4)

$$nu^{n-1}(t)u'(t)=nf(t)x^m(\sigma_1(t))\int_0^t g(\tau)W(x^r(\sigma_2(\tau))d\tau+h(t)x^l(\sigma_3(t)), t\in[0,\infty).$$

Now, if $0 \leq \sigma_1(t) \leq t$, we have

$$\begin{aligned}
x^n(\sigma_1(t)) &< u^n(\sigma_1(t)) \\
&= (c+\varepsilon)^n \\
&\quad + n\int_0^{\sigma_1(t)} \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau\right)+h(s)x^l(\sigma_3(s))\right]ds \\
&\leq (c+\varepsilon)^n \\
&\quad + n\int_0^t \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau\right)+h(s)x^l(\sigma_3(s))\right]ds \\
&= u^n(t).
\end{aligned}$$

Also, if $a \leq \sigma_1(t) \leq 0$, we have

$$x(\sigma_1(t))=\psi(\sigma_1(t)) < c+\varepsilon < u(t).$$

Thus, in any case we have

$$x(\sigma_1(t)) < u(t), t\in[0,\infty).$$

Similarly, we have also

$$x(\sigma_i(t)) < u(t), t \in [0,\infty), i = 2,3.$$

By (4), since $0 < m, l \leq n - r$, we have

$$u^{n-1}(t)u'(t) < f(t)u^{n-r}(t)\int_0^t g(\tau)W(u^r(\tau))d\tau + u^{n-r}(t)h(t), t \in [0,\infty).$$

or

$$u^{r-1}(t)u'(t) < f(t)\int_0^t g(\tau)W(u^r)(\tau)d\tau + h(t), t \in [0,\infty).$$

Integrating both sides from 0 to t we have

$$u^r(t) < p(t) + r\int_0^t f(s)\left(\int_0^s g(\tau)W(u^r(\tau))d\tau\right)ds, t \in [0,\infty),$$

where

$$p(t) = (c+\varepsilon)^r + r\int_0^t h(s)ds, t \in [0,\infty).$$

For an arbritrary $T \in [0,\infty)$ we have

$$u^r(t) < p(T) + r\int_0^t f(s)\left(\int_0^s g(\tau)W(u^r(\tau))d\tau\right)ds, t \in [0,\infty).$$

We set

$$v(t) = p(T) + r\int_0^t f(s)\left(\int_0^s g(\tau)W(u^r(\tau))d\tau\right)ds, t \in [0,\infty).$$

Since $u^r(t) < v(t), \quad t \in [0,T]$ and W is nondecreasing, we have

$$v'(t) \leq rf(t)W(v(t))\int_0^t g(\tau)d\tau, t \in [0,T].$$

Thus

$$\frac{d}{dt}G(v(t)) \leq rf(t)\int_0^t g(\tau)d\tau, t \in [0,T].$$

Integrating both sides from 0 to T, we have

$$G(v(T)) \leq G(p(T)) + r\int_0^T f(s)(\int_0^s g(\tau)d\tau)ds.$$

Hence

$$v(T) \leq G^{-1}\left[G(p(T)) + r\int_0^T f(s)\left(\int_0^s g(\tau)d\tau\right)ds\right].$$

Since T is arbitrary and $x(t) < u(t), \quad t \in [0, \infty)$, the result is obvious by letting $\varepsilon \to 0$.

(iii) We define a function u: $[0, \infty) \to [0, \infty)$, by

$$u^n(t) = (c+\varepsilon)^n + n\int_0^t \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(logx(\sigma_2(\tau)))d\tau\right) + h(s)x^l(\sigma_3(s))\right] ds,$$

where $\varepsilon > 0$, is an arbitrary constant. Then

$$u(0) = c + \varepsilon$$

and for every $t \in [0, \infty)$ we have

$$nu^{n-1}(t)u'(t) = nf(t)x^m(\sigma_1(t)) \int_0^t g(\tau)W(logx(\sigma_2(\tau)))d\tau + h(t)x^l(\sigma_3(t)). \tag{5}$$

As in the proof of step (i) and, since $0 < m, l \leq n$, we can prove that

$$x^m(\sigma_i(t)) < u^n(t) \quad \text{and} \quad x^l(\sigma_i(t)) < u^n(t), \quad t \in [0, \infty), i = 1, 2, 3.$$

Hence, by (5) we have

$$u^{n-1}(t)u'(t) < f(t)u^n(t) \int_0^t g(\tau)W(logu(\tau))d\tau + h(t)u^n(t), t \in [0, \infty).$$

or

$$\frac{u'(t)}{u(t)} \leq f(t) \int_0^t g(\tau)W(logu(\tau))d\tau + h(t), t \in [0, \infty).$$

Integrating both sides from 0 to t, we have

$$logu(t) \leq \hat{p}(t) + \int_0^t f(s)(\int_0^s g(\tau)W(logu(\tau))d\tau)ds, t \in [0, \infty),$$

where

$$\hat{p}(t) = log(c + \varepsilon) + \int_0^t h(s)ds.$$

We omit the rest of the proof since it is similar to that in the above step (i). □

References

[1] B.G.Pachpate, *A note on certain integral inequalities with delay*, Periodica Math. Hungarica **31** (1995), 99–102.

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